Regional Spatial Use with Autonomous Vehicles

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Abstract

The introduction of autonomous vehicles will reshape regional land use and populations. A bottleneck traffic model and monocentric city model combine to show how cities will evolve in the presence of both autonomous and standard vehicles. Autonomous vehicles will cost more, but the time spent commuting will be more pleasant and the vehicles will flow more easily through the bottleneck. Among the many results of the model, as the autonomous vehicle advantage in flow rate increases, the measure of drivers will increase though they will live in a smaller area and the measure of autonomous car users will increase and the size of the commuting region will expand.

Keywords: Autonomous Vehicles, Congestion, Spatial Competition, Regional Equilibrium, Residential Location

JEL: O33, R12, R13, R23, R41

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1. Introduction

Interest in the role autonomous vehicles will have on shaping cities has reached the popular press.[1] City and regional planners are considering how to adapt and manage long lead time and durable infrastructure projects to accommodate autonomous vehicles. Private firms are also considering the role of changing technology when planning investment.

How changes to spatial use patterns will occur is an important question for regional and urban planners. A reduced model of real world transportation with conservative assumptions about changes in travel modes generates interesting dynamics. This paper combines models of land use and traffic congestion to examine urban area and population changes in a setting with both autonomous and driven vehicles.

This paper merges the bottleneck model of the commute formulated by Vickrey (1969) and extended by van den Berg and Verhoef (2016) to include both driven and autonomous vehicles with the Alonso-Muth-Mills[2] monocentric city model particularly of the form used by Wheaton (1974). The bottleneck model assumes commuters travel to a central business district (CBD) at a preferred common time. The flow is constrained by a bottleneck,

[1]For examples see Higgins, Tim "What Driverless Cars Will Bring to Cities" *The Wall Street Journal* 27 Jun. 2018 or Shaver, Katherine "City Planners Eye Self-Driving Vehicles to Correct Mistakes of the 20th-Century Auto" *The Washington Post* 20 Jul. 2019 accessed by `https://www.washingtonpost.com/transportation/2019/07/20/city-planners-eye-self-driving-vehicles-correct-mistakes-th-century-auto` on 31 Jul 2019.

[2]The seminal works in this literature are Alonso (1964), Muth (1961) and Mills (1967).

which is a place of limited capacity which all commuters must go through. The desire to arrive at the same time and the limited traffic capacity generates a period of time when there is congestion analogous to the commonly experienced rush hour. The Alonso-Muth-Mills model assumes a CBD and commuters who choose how far to live from the CBD, how much housing to use given profit maximizing landlords and how much of a composite consumption good to use.

The model presented here is both reduced from the real world and conservative about the assumptions that it makes for the future forms of transportation. It is simplified by assuming homogeneous preferences for each type of commuter and homogeneous income for all residents. Other simplifying assumptions are that all individuals must commute and travel must take place by vehicle. The approach is conservative in that it does not imagine massive changes in the pattern of vehicle ownership.

Burns (2013) is among many papers that envision autonomous vehicles causing a large increase in car sharing or that automobiles will no longer be held by private individuals. Although a formal argument against that position is beyond the scope of this paper, a few brief comments about why that position is unpersuasive are presented here. In a car sharing scheme, the vehicle becomes a common pool resource and monitoring to prevent negative externalities such as damage to the interior or leaving trash in the vehicle will be difficult. Beyond visible externalities, which may be subject to a Coasean bargain, private ownership guards against the transmission of communicable diseases. Additionally, having a private vehicle available on short notice is valuable. This may be achievable with car sharing in a dense urban area, but is more difficult in a less dense suburban setting. Finally, being able to customize a vehicle and use it for storage provides value to the owner. For these reasons, this paper will assume private ownership of autonomous vehicles.

There have been a few prior works that combine the bottleneck and monocentric city with only driven vehicles. The closest to this study is Fosgerau et al. (2018) which examines a single type of commuter in a bottleneck model and monocentric city. That paper is particularly interested in optimal tolling, an issue which is not addressed in this paper. A related paper is Gubins and Verhoef (2014) which uses a Cobb-Douglas utility function and considers a utility from time spent at home. The paper also considers tolling and only driven vehicles. In contrast, this paper has less restrictive assumptions on the utility function and allows for autonomous vehicles. Takayama and Kuwahara (2017) considers a monocentric city with a bottleneck commuting technology with heterogeneous commuters with quasi-linear utility. The commuter types differ in time costs and income, though the paper does find commuters choose location based on their type, similar to a finding in this paper.

A few studies of autonomous vehicles and urban land use have also been produced. Rappaport (2016) constructs a detailed calibrated model of urban space use and congestion. In several scenarios the paper considers autonomous vehicles replacing driven vehicles. Zakharenko (2016) examines a model with autonomous vehicles that emphasizes parking. The nature of the model is quite different, however, in that the centralization is driven by the need to access a port to export and import goods and congestion is not considered.

The rest of the paper is organized as follows. Section 2 presents the model. First in section 2.1 the commuting portion of the model is presented. Then in section 2.2, the model is placed in the monocentric city. Section 3 briefly describes the approach to find the main results, with subsections dedicated to sets of statics. Section 4 concludes the paper.

2. Model

The city is arranged as a variation of the monocentric city models of Alonso (1964), Muth (1961), Mills (1967) and Wheaton (1974). Individuals commute to the central business district (CBD) and earn a wage, y, with which they rent housing, q, purchase a consumption good, c, and pay for their commute. The commute is a variation of the bottleneck model of Vickrey (1969) formulated to include autonomous vehicles by van den Berg and Verhoef (2016). In addition, the cost of a standard driven car is normalized to zero and the difference between the cost of an autonomous car and a driven car, $l > 0$, is exogenous.

2.1. The Commute

Commuters prefer to arrive at the CBD at time t_W. Commuters who arrive late pay a penalty, γ, for each unit of time they are late. Commuters who arrive early pay a penalty per unit of time early of β. Driving costs α per unit of time. Compared to driving, autonomous commuting allows time and effort spent concentrating on the road to be used for other tasks. Therefore, the cost of commuting in an autonomous car is lower than the cost of driving by the multiplier θ, with $0 < \theta < 1$, so the cost is $\alpha\theta$ per unit of time. In addition, autonomous cars take up less space at the bottleneck, so the capacity is m driven vehicles and λm autonomous vehicles where $\lambda > 0$.

At the time of the commute, the location decision is exogenous for individuals. However, consumers will consider the commuting solution when choosing a location. Individuals who drive and those who commute in autonomous vehicles minimize their commuting costs given their housing location, x, by choosing a time, t, to leave their homes respectively

$$\begin{aligned} &\min_t \alpha x + \alpha(\tau(t) - t) + \max(\beta(t_W - \tau(t)), \gamma(\tau(t) - t_W)) \\ &\min_t \alpha\theta x + \alpha\theta(\tau(t) - t) + \max(\beta(t_W - \tau(t)), \gamma(\tau(t) - t_W)) \end{aligned} \tag{1}$$

where $\tau(t)$ is the congestion cost of arriving at the bottleneck at time t.

In equilibrium, individuals in each group are indifferent about the times they arrive at the bottleneck. The equilibrium congestion costs for each of the N_d drivers and N_a commuters in autonomous vehicles are respectively

$$\begin{aligned} C_d &= \eta(\frac{N_d}{m} + \frac{N_a}{\lambda m}) \\ C_a &= \eta(\frac{\theta N_d}{m} + \frac{N_a}{\lambda m}) \end{aligned} \tag{2}$$

where $\eta = \frac{\beta\gamma}{\beta+\gamma}$. This time preference parameter related to the relative costs of being early to work and being late is exogenous to the location decisions of commuters and will be treated as a constant in the rest of the paper. The parameter is determined both by commuters' intrinsic preferences and their jobs' demands, so policy makers have no influence over it. The solution is similar to the solution in van den Berg and Verhoef (2016) except this paper considers the measure of each type of commuters rather than the share.

2.2. The City

The monocentric city has an population of residents of measure N, N_a with autonomous vehicles and N_d who drive. Individuals choose a location to live, x, and a quantity of housing, q, at the market rent $r(x)$. The market rent at location x is the maximum rent from the bid-rent schedules, $r_d(x)$ and $r_a(x)$, for driven and autonomous vehicles respectively. Individuals also choose the level of consumption, c, at the market price normalized to one. Much of the following structure is borrowed from the approach of Wheaton (1974) with adjustments made to accommodate the introduction of autonomous vehicles and transportation mode choice.

All individuals share common preferences for consumption goods and housing expressed in the utility function $U(c, q)$. The utility function is assumed to be strictly quasiconcave with both the consumption and housing goods normal. In equilibrium, all individuals obtain the same level of utility,

$$U(c, q) = \hat{u}. \tag{3}$$

Budget constraints differ depending on the transportation mode. In addition to the difference in per time costs, a fixed cost differential of $l > 0$ is assumed to be required to own an autonomous vehicle. This cost covers the additional sensors and computing power requried for a vehicle to navigate itself. Those who drive and those who use autonomous cars have budget constraints respectively

$$\begin{aligned} y &= C_d + \alpha x + r_d(x)q + c \\ y &= C_a + \alpha\theta x + r_a(x)q + c + l. \end{aligned} \tag{4}$$

The housing suppliers will rent to the group with the higher willingness to pay. Let the distance from the CBD where the bid-rent functions, $r_d(x)$ and $r_a(x)$, are equal be $\tilde{x}$.

Claim 2.1. *For distances closer to the CBD than $\tilde{x}$, commuters drive. For distances farther from the CBD, commuters choose autonomous vehicles.*

The intuition is that a lower cost per time spent commuting in an autonomous vehicle makes commuting from farther distances compensate for the fixed cost of the vehicle. This sharp demarcation is consistent with similar models. Heterogenity of preferences and work locations would lead to a less pronounced boundary in the real world, but the pattern that individuals would benefit more from autonomous vehicles during longer commutes and so would be more willing to pay for an autonomous vehicle is likely to hold. A proof is provided in Appendix A.

At $\tilde{x}$ the two types of commuter will consume the same bundle of $\{c, q\}$. So

$$\begin{aligned} r_a(\tilde{x}) &= r_d(\tilde{x}) \\ \frac{y - C_a - \alpha\theta\tilde{x} - c - l}{q} &= \frac{y - C_d - \alpha\tilde{x} - c}{q} \\ \tilde{x} &= \frac{l}{\alpha(1-\theta)} - \frac{\eta}{\alpha m} N_d. \end{aligned} \tag{5}$$

Landlords maximize the rent they charge subject to the consumer's budget constraints and the equilibrium condition (3). This leads to rent curves for each type that equate the marginal utlities and the budget constraint.

$$\frac{\partial u/\partial q}{\partial u/\partial c} = R \equiv \begin{cases} \frac{y - C_d - \alpha x - c}{q} & \text{for } 0 < x \leq \tilde{x} \\ \frac{y - C_a - \alpha\theta x - c - l}{q} & \text{for } x > \tilde{x}. \end{cases} \tag{6}$$

The consumption, quantity of housing and rent function can then be expressed as functions of the price difference of autonomous vehicles, the distance from the CBD, the utility level and the income:

$$\begin{aligned} c &= c(x, l, \alpha, \theta, \lambda, m, N_d, N_a, u, y) \\ q &= q(x, l, \alpha, \theta, \lambda, m, N_d, N_a, u, y) \\ r &= r(x, l, \alpha, \theta, \lambda, m, N_d, N_a, u, y). \end{aligned} \tag{7}$$

All land has an alternative agricultural value of A. So the boundary of the city is set at $\hat{x}$ where

$$r(\hat{x}, l, \alpha, \theta, \lambda, m, N_d, N_a, u, y) = A. \tag{8}$$

All individuals are contained within the boundary of the city. The N_d individuals who drive are contained within $\tilde{x}$ and the N_a autonomous commuters are located between $\tilde{x}$ and $\hat{x}$. The inverse of the quantity of housing per individual is the radial density and integrating the circumference along the radius for each type of commuter gives their measures:

$$2\pi \int_0^{\tilde{x}} \frac{t}{q(t, l, \alpha, \theta, \lambda, m, N_d, N_a, u, y)} \mathrm{d}t = N_d \tag{9}$$

$$2\pi \int_{\tilde{x}}^{\hat{x}} \frac{t}{q(t, l, \alpha, \theta, \lambda, m, N_d, N_a, u, y)} \mathrm{d}t = N_a. \tag{10}$$

3. Statics

This section describes the statics for a small metro in the sense that the local changes considered do not change the level of national utility. This is analogous to small country trade models. A brief discussion of the solution method is followed by descriptions of individual statics.

The solution method borrows from Wheaton (1974). First a system of equations formed by the total differentials of the utility condition, (3), and the maximized rents, (6), gives the statics for q and c. Second, applying the Envelope Theorem to the maximized rents gives partials for R. Finally, a system of equations is formed from the total differentials of the boundary conditions of $\tilde{x}$ and $\hat{x}$, (5) and (8) respectively, as well as the population conditions for commuters who drive and those who use autonomous vehicles, (9) and (10) respectively. Using results from the first two steps, many of the statics derived in the third step for $\tilde{x}$, N_d, N_a and $\hat{x}$ are signed. More details are presented in Appendix B. All claims in this section follow the logic of Appendix B, but are presented more fully in Appendix C.3.

3.1. Change in the Fixed Premium for Autonomous Vehicles

Consider a per vehicle tax on the purchase of autonomous vehicles imposed on the metro area.[3] This tax increases the fixed cost of the autonomous vehicles, l.

The metro area, $\hat{x}$, shrinks in total though the distance containing commuters who drive, $\tilde{x}$, expands. There is a total population, N, decline because the decline in the measure of autonomous commuters, N_a, is larger than the increase in the measure of commuters who drive, N_d.

[3] A rebate may be a more likely policy proposal, in which case the signs would be reversed.

Claim 3.1.

$$\frac{d\tilde{x}}{dl} > 0 \qquad \frac{d\hat{x}}{dl} < 0$$
$$\frac{dN_d}{dl} > 0 \qquad \frac{dN_a}{dl} < 0 \qquad \frac{dN}{dl} < 0$$

3.2. Change in the Capacity of the Bottleneck

Increasing the capacity of the road system by adding lanes to combat traffic does not lead to many clear predictions. The overall population and area of the city's regions are indeterminate. The only concrete prediction is that the measure of people who drive will increase. An increase in driven vehicles may crowd out the autonomous vehicles, while the increase in capacity encourages autonomous vehicles. The net effect on autonomous vehicles is not determined without further restrictions on the model.

Claim 3.2.

$$\frac{d\tilde{x}}{dm} \lesseqgtr 0 \qquad \frac{d\hat{x}}{dm} \lesseqgtr 0$$
$$\frac{dN_d}{dm} > 0 \qquad \frac{dN_a}{dm} \lesseqgtr 0 \qquad \frac{dN}{dm} \lesseqgtr 0$$

3.3. Change in the Capacity Advantage of Autonomous Vehicles

While a general increase in capacity leads to few predictions, increasing the capacity advantage of autonomous vehicles has clear and specific predictions. Policies that might encourage this change include decreasing required safety margins for autonomous vehicles and improved vehicle-to-vehicle and vehicle-to-infrastructure communication.

The measure of both types of commuters increases so the overall population increases. The area where drivers live shrinks while the metro boundary expands.

Claim 3.3.

$$\frac{d\tilde{x}}{d\lambda} < 0 \qquad \frac{d\hat{x}}{d\lambda} > 0$$
$$\frac{dN_d}{d\lambda} > 0 \qquad \frac{dN_a}{d\lambda} > 0 \qquad \frac{dN}{d\lambda} > 0$$

3.4. Change in the Time Cost Advantage of Autonomous Vehicles

Recalling that θ is the multiplier of time cost in an autonomous vehicle and $0 < \theta < 1$, considering a decrease in θ is likely more relevant. As manufacturers learn about consumer preferences for autonomous vehicle amenities, θ is likely to decrease. A policy designed to alter θ is unlikely because of the difficulty in monitoring time spent in vehicles, so changes in θ are more likely to occur as autonomous vehicles mature as a technology.

As θ decreases, the number of commuters who drive will decrease and the number of commuters in autonomous will increase. The net effect will be an increase in the population in the metro area. The size of the metro area is indeterminate, but the size of the ring of commuters who drive will shrink.

Claim 3.4.

$$\frac{d\tilde{x}}{d\theta} < 0 \qquad \frac{d\hat{x}}{d\theta} \lesseqgtr 0$$
$$\frac{dN_d}{d\theta} > 0 \qquad \frac{dN_a}{d\theta} < 0 \qquad \frac{dN}{d\theta} < 0$$

3.5. Change in Agricultural Land Value

The final consideration in this paper is a change in the outside use value of land. An increase in the agricultural land value shrinks the border of the city and shrinks the internal border dividing autonomous and driving commuters. The total population of the city declines, though the population of driving commuters increases.

Claim 3.5.

$$\frac{d\tilde{x}}{dA} < 0 \qquad \frac{d\hat{x}}{dA} < 0$$
$$\frac{dN_d}{dA} > 0 \qquad \frac{dN_a}{dA} < 0 \qquad \frac{dN}{dA} < 0$$

4. Conclusion

This paper builds a rich set of statics useful for urban and regional planners as autonomous vehicles are introduced to the transportation market. The statics are derived from a model of the interaction between owners of autonomous vehicles and driven vehicles in a monocentric city framework with bottleneck congestion. Despite several important simplifying assumptions in monocentric city models, the statics developed from them appear to hold.[4]

As the price difference between driven vehicles and autonomous vehicles shrinks, whether through subsidies or economies of scale in autonomous car production, the region will experience both population growth and a growth in area. Also, as the benefit of time spent in an autonomous vehicle grows, the population of a region will increase, but the size of the region is indeterminate. Expanding the capacity of the bottleneck may change both population and area of the region in either direction. However, improving the capacity advantage of autonomous vehicles will grow the population and the area of the region.

Extending these models using reasonable assumptions about the behavior of autonomous vehicles gives regional and urban planners a foundation for

[4]See McMillen (2007) for an overview.

evaluating potential policy decisions. Future work deriving from this paper could examine alternative assumptions, congestion pricing strategies and add alternative transportation schemes.

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Declarations of interest: none

References

Alonso, W., 1964. Location and land use: Toward a General Theory of Land Rent. Harvard University Press, Cambridge, MA.

van den Berg, V.A., Verhoef, E., 2016. Autonomous cars and dynamic bottleneck congestion: The effects on capacity, value of time and preference heterogeneity. Transportation Research Part B: Methodological 94, 43–60. doi:10.1016/j.trb.2016.08.018.

Burns, L.D., 2013. Sustainable mobility: a vision of our transport future. Nature 497, 181–182. doi:10.1038/497181a.

Fosgerau, M., Kim, J., Ranjan, A., 2018. Vickrey meets alonso: Commute scheduling and congestion in a monocentric city. Journal of Urban Economics 105, 40–53. doi:10.1016/j.jue.2018.02.003.

Gubins, S., Verhoef, E.T., 2014. Dynamic bottleneck congestion and residential land use in the monocentric city. Journal of Urban Economics 80, 51–61. doi:10.1016/j.jue.2013.09.001.

McMillen, D.P., 2007. Testing for Monocentricity. John Wiley & Sons, Ltd. chapter 8. pp. 128–140. doi:10.1002/9780470996225.ch8.

Mills, E.S., 1967. An aggregative model of resource allocation in a metropolitan area. The American Economic Review 57, 197–210.

Muth, R.F., 1961. The Spatial Structure Of The Housing Market. Papers in Regional Science 7, 207–220. doi:10.1111/j.1435-5597.1961.

Rappaport, J., 2016. Productivity, congested commuting, and metro size. Research Working Paper RWP 16-3. Federal Reserve Bank of Kansas City. doi:10.18651/RWP2016-03.

Takayama, Y., Kuwahara, M., 2017. Bottleneck congestion and residential location of heterogeneous commuters. Journal of Urban Economics 100, 65–79. doi:10.1016/j.jue.2017.05.001.

Vickrey, W.S., 1969. Congestion theory and transport investment. The American Economic Review 59, 251–260.

Wheaton, W.C., 1974. A comparative static analysis of urban spatial structure. Journal of Economic Theory 9, 223–237.

Zakharenko, R., 2016. Self-driving cars will change cities. Regional Science and Urban Economics 61, 26–37. doi:10.1016/j.regsciurbeco.2016.09.003.

Appendicies

Appendix A.

Proof. Claim 2.1

By the definition of $\tilde{x}$, the total costs of both modes of transportation are equal, that is

$$C_a + \alpha\theta\tilde{x} + l - (C_d + \alpha\tilde{x}) = 0$$

First demonstrate that distances farther from the CBD autonomous vehicles are less costly. Let $\epsilon > 0$. Subtracting the total costs of driving from the total costs of autonmous vehicles and using the above expression

$$C_a + \alpha\theta(\tilde{x} + \epsilon) + l - (C_d + \alpha(\tilde{x} + \epsilon)) = \\ (\theta - 1)\alpha\epsilon < 0$$

So the costs of autonomous vehicles are lower beyond for $x > \tilde{x}$.

The proof that distances are closer to the CBD are less costly for drivers is the same except let $\epsilon < 0$ the reverse inequality reverses. □

Appendix B.

Appendix B.1. Step One

Form a systems of equations with the total differentials of the utility condition, (3), and the maximized rents, (6). This is two systems of equations, one for drivers, who live inside $\tilde{x}$ and another for users of autonomous vehicles who live in the ring outside $\tilde{x}$ and inside the urban boundary, $\hat{x}$. From this solve the statics for the quantity of housing, q and the consumption, c.

For housing in the circle of driving commuters, that is locations x where $0 < x \leq \tilde{x}$, the system of equations is as follows.

$$\begin{bmatrix} \frac{\partial u}{\partial q} & \frac{\partial u}{\partial c} \\ \frac{\partial R}{\partial q} + \frac{R}{q} & \frac{\partial R}{\partial c} + \frac{1}{q} \end{bmatrix} \begin{bmatrix} dq \\ dc \end{bmatrix} =$$
$$\begin{bmatrix} du \\ \frac{1}{q}(dy - \alpha dx - x d\alpha - \eta(\frac{dN_d}{m} + \frac{dN_a}{\lambda m} - \frac{N_a d\lambda}{\lambda^2 m} - \frac{(N_a + \lambda N_d)dm}{\lambda m^2})) \end{bmatrix}$$

For housing in the ring of autonomous commuters, that is for locations x where $\tilde{x} < x \leq \hat{x}$, the system of equations is as follows.

$$\begin{bmatrix} \frac{\partial u}{\partial q} & \frac{\partial u}{\partial c} \\ \frac{\partial R}{\partial q} + \frac{R}{q} & \frac{\partial R}{\partial c} + \frac{1}{q} \end{bmatrix} \begin{bmatrix} dq \\ dc \end{bmatrix} =$$
$$\begin{bmatrix} du \\ \frac{1}{q}(dy - \alpha\theta dx - \theta x d\alpha - \alpha x d\theta - dl - \eta(\frac{\theta dN_d + N_d d\theta}{m} + \frac{dN_a}{\lambda m} - \frac{N_a d\lambda}{\lambda^2 m} - \frac{(N_a + \theta\lambda N_d)dm}{\lambda m^2})) \end{bmatrix}$$

Using Theorem 1 from Wheaton (1974), reprinted below, statics can be caluclated and signed.

Theorem 1. *Wheaton (1974) Theorem 1*
If utility is strictly quasiconcave, and if both goods have positive income effects, then $\frac{\partial R}{\partial q} < 0$, $\frac{\partial R}{\partial c} > 0$, *where* $R = \frac{\partial u}{\partial q} / \frac{\partial u}{\partial c}$.

The statics are presented in Appendix C.1.

Appendix B.2. Step Two

Apply the Envelope Theorem to the maximized rents, (6). This generates partial derivatives related to the rent functions. The partials are reported in Appendix C.2.

Appendix B.3. Step Three

Form a system of differential equations from the boundary conditions, (5) and (8), as well as the population conditions, (9) and (10). The system of equations is presented below.

$$\begin{bmatrix} 1 & \frac{\eta}{\alpha m} & 0 & 0 \\ 0 & \frac{\partial R}{\partial N_d} & \frac{\partial R}{\partial N_a} & \frac{\partial R}{\partial \hat{x}} \\ -\frac{\tilde{x}}{q(\tilde{x};\cdot)} & \frac{1}{2\pi} + \int_0^{\tilde{x}} \frac{t}{q^2(t;\cdot)} \frac{\partial q}{\partial N_d} dt & \int_0^{\tilde{x}} \frac{t}{q^2(t;\cdot)} \frac{\partial q}{\partial N_a} dt & 0 \\ \frac{\tilde{x}}{q(\tilde{x};\cdot)} & \int_{\tilde{x}}^{\hat{x}} \frac{t}{q^2(t;\cdot)} \frac{\partial q}{\partial N_d} dt & \frac{1}{2\pi} + \int_{\tilde{x}}^{\hat{x}} \frac{t}{q^2(t;\cdot)} \frac{\partial q}{\partial N_a} dt & -\frac{\hat{x}}{q(\hat{x};\cdot)} \end{bmatrix} \begin{bmatrix} d\tilde{x} \\ dN_d \\ dN_a \\ d\hat{x} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{\alpha(1-\theta)} dl - \frac{\tilde{x}}{\alpha} d\alpha + \frac{l}{\alpha(1-\theta)^2} d\theta + \frac{\eta N_d}{\alpha m^2} dm \\ dA - (\frac{\partial R}{\partial l} dl + \frac{\partial R}{\partial \alpha} d\alpha + \frac{\partial R}{\partial \theta} d\theta + \frac{\partial R}{\partial \lambda} d\lambda + \frac{\partial R}{\partial m} dm + \frac{\partial R}{\partial u} du + \frac{\partial R}{\partial y} dy) \\ -\int_0^{\tilde{x}} \frac{t}{q^2(t;\cdot)} (\frac{\partial q}{\partial l} dl + \frac{\partial q}{\partial \alpha} d\alpha + \frac{\partial q}{\partial \theta} d\theta + \frac{\partial q}{\partial \lambda} d\lambda + \frac{\partial q}{\partial m} dm + \frac{\partial q}{\partial u} du + \frac{\partial q}{\partial y} dy) dt \\ -\int_{\tilde{x}}^{\hat{x}} \frac{t}{q^2(t;\cdot)} (\frac{\partial q}{\partial l} dl + \frac{\partial q}{\partial \alpha} d\alpha + \frac{\partial q}{\partial \theta} d\theta + \frac{\partial q}{\partial \lambda} d\lambda + \frac{\partial q}{\partial m} dm + \frac{\partial q}{\partial u} du + \frac{\partial q}{\partial y} dy) dt \end{bmatrix} \tag{Appendix B.1}$$

By making appropriate substitutions from the results of Step One and Step Two, many of the statics can be signed.

Appendix C.

Appendix C.1. Statics from Step One

$$\frac{dq}{du} = \frac{\frac{\partial u}{\partial c}(\frac{\partial R}{\partial c} + \frac{1}{q})}{R\frac{\partial R}{\partial c} - \frac{\partial R}{\partial q}} > 0 \qquad \frac{dc}{du} = -\frac{\frac{\partial u}{\partial c}(\frac{\partial R}{\partial q} + \frac{R}{q})}{R\frac{\partial R}{\partial c} - \frac{\partial R}{\partial q}} \lesseqgtr 0$$
$$\frac{dq}{dy} = -\frac{1}{q(R\frac{\partial R}{\partial c} - \frac{\partial R}{\partial q})} < 0 \qquad \frac{dc}{dy} = -R\frac{dq}{dy} > 0$$

$$\frac{dq}{d\lambda} = \frac{-\eta N_a}{\lambda^2 mq(R\frac{\partial R}{\partial c} - \frac{\partial R}{\partial q})} < 0 \qquad \frac{dc}{d\lambda} = -R\frac{dq}{d\lambda} > 0$$

$$\frac{dq}{d\alpha} = \begin{cases} \dfrac{x}{q(R\frac{\partial R}{\partial c} - \frac{\partial R}{\partial q})} > 0 & \text{if } 0 < x < \tilde{x} \\ \dfrac{\theta x}{q(R\frac{\partial R}{\partial c} - \frac{\partial R}{\partial q})} > 0 & \text{if } x > \tilde{x} \end{cases} \qquad \frac{dc}{d\alpha} = -R\frac{dq}{d\alpha} < 0$$

$$\frac{dq}{dx} = \begin{cases} \dfrac{\alpha}{q(R\frac{\partial R}{\partial c} - \frac{\partial R}{\partial q})} > 0 & \text{if } 0 < x < \tilde{x} \\ \dfrac{\alpha\theta}{q(R\frac{\partial R}{\partial c} - \frac{\partial R}{\partial q})} > 0 & \text{if } x > \tilde{x} \end{cases} \qquad \frac{dc}{dx} = -R\frac{dq}{dx} < 0$$

$$\frac{dq}{dm} = \begin{cases} \dfrac{-\eta(N_a + \lambda N_d)}{\lambda m^2 q(R\frac{\partial R}{\partial c} - \frac{\partial R}{\partial q})} < 0 & \text{if } 0 < x < \tilde{x} \\ \dfrac{-\eta(N_a + \theta\lambda N_d)}{\lambda m^2 q(R\frac{\partial R}{\partial c} - \frac{\partial R}{\partial q})} < 0 & \text{if } x > \tilde{x} \end{cases} \qquad \frac{dc}{dm} = -R\frac{dq}{dx} > 0$$

$$\frac{dq}{d\theta} = \begin{cases} 0 & \text{if } 0 < x < \tilde{x} \\ \dfrac{\alpha x + \eta\frac{N_d}{m}}{q(R\frac{\partial R}{\partial c} - \frac{\partial R}{\partial q})} > 0 & \text{if } x > \tilde{x} \end{cases} \qquad \frac{dc}{d\theta} = \begin{cases} 0 & \text{if } 0 < x < \tilde{x} \\ \dfrac{-R(\alpha x + \eta\frac{N_d}{m})}{q(R\frac{\partial R}{\partial c} - \frac{\partial R}{\partial q})} < 0 & \text{if } x > \tilde{x} \end{cases}$$

$$\frac{dq}{dl} = \begin{cases} 0 & \text{if } 0 < x < \tilde{x} \\ \dfrac{1}{q(R\frac{\partial R}{\partial c} - \frac{\partial R}{\partial q})} > 0 & \text{if } x > \tilde{x} \end{cases} \qquad \frac{dc}{dl} = \begin{cases} 0 & \text{if } 0 < x < \tilde{x} \\ \dfrac{-R}{q(R\frac{\partial R}{\partial c} - \frac{\partial R}{\partial q})} < 0 & \text{if } x > \tilde{x} \end{cases}$$

$$\frac{dq}{dN_d} = \begin{cases} \dfrac{\eta}{mq(R\frac{\partial R}{\partial c} - \frac{\partial R}{\partial q})} > 0 & \text{if } 0 < x < \tilde{x} \\ \dfrac{\eta\theta}{mq(R\frac{\partial R}{\partial c} - \frac{\partial R}{\partial q})} > 0 & \text{if } x > \tilde{x} \end{cases} \qquad \frac{dc}{dN_d} = -R\frac{dq}{dN_d} < 0$$

$$\frac{dq}{dN_a} = \frac{\eta}{\lambda mq(R\frac{\partial R}{\partial c} - \frac{\partial R}{\partial q})} > 0 \qquad \frac{dc}{dN_a} = -R\frac{dq}{dN_a} < 0$$

Appendix C.2. Partials from Step Two

$$\frac{\partial R}{\partial m} = \begin{cases} \frac{\eta}{q}(\frac{N_d}{m^2} + \frac{N_a}{\lambda m^2}) > 0 & \text{if } 0 < x < \tilde{x} \\ \frac{\eta}{q}(\frac{\theta N_d}{m^2} + \frac{N_a}{\lambda m^2}) > 0 & \text{if } x > \tilde{x} \end{cases} \qquad \frac{\partial R}{\partial \lambda} = \frac{\eta N_a}{\lambda^2 mq} > 0$$

$$\frac{\partial R}{\partial N_d} = \begin{cases} -\frac{\eta}{mq} < 0 & \text{if } 0 < x < \tilde{x} \\ -\frac{\eta\theta}{mq} < 0 & \text{if } x > \tilde{x} \end{cases} \qquad \frac{\partial R}{\partial N_a} = -\frac{\eta}{\lambda mq} < 0$$

$$\frac{\partial R}{\partial \theta} = \begin{cases} 0 & \text{if } 0 < x < \tilde{x} \\ -\frac{1}{q}(\frac{\eta N_d}{m} + \alpha) < 0 & \text{if } x > \tilde{x} \end{cases} \qquad \frac{\partial R}{\partial y} = \frac{1}{q} > 0$$

$$\frac{\partial R}{\partial \alpha} = \begin{cases} -\frac{x}{q} < 0 & \text{if } 0 < x < \tilde{x} \\ -\frac{\theta x}{q} < 0 & \text{if } x > \tilde{x} \end{cases} \qquad \frac{\partial R}{\partial x} = \begin{cases} -\frac{\alpha}{q} < 0 & \text{if } 0 < x < \tilde{x} \\ -\frac{\alpha\theta}{q} < 0 & \text{if } x > \tilde{x} \end{cases}$$

$$\frac{\partial R}{\partial l} = \begin{cases} 0 & \text{if } 0 < x < \tilde{x} \\ -\frac{1}{q} & \text{if } x > \tilde{x} \end{cases} \qquad \frac{\partial R}{\partial u} = -\frac{1}{q}(R\frac{dq}{du} + \frac{dc}{du}) = -\frac{\partial u/\partial c}{q} < 0$$

Appendix C.3. Claims from Section 3

Notice from Appendix Appendix C.1 that

$$\frac{dq}{dN_d} = \lambda \frac{dq}{dN_a} \qquad \text{if } 0 < x < \tilde{x}$$
$$\frac{dq}{dN_d} = \theta\lambda \frac{dq}{dN_a} \qquad \text{if } \tilde{x} < x$$

For clarity, define $p_1 \equiv \int_0^{\tilde{x}} \frac{t}{q^2(t;\cdot)} \frac{\partial q}{\partial N_a} dt$ and $p_2 \equiv \int_{\tilde{x}}^{\hat{x}} \frac{t}{q^2(t;\cdot)} \frac{\partial q}{\partial N_a} dt$. Notice from Appendix Appendix C.1 and the nature of the bounds, $p_1 > 0$ and $p_2 > 0$. With these substitutions and those from Appendix Appendix C.2, equation (Appendix B.1) can be written as

$$\begin{bmatrix} 1 & \frac{\eta}{\alpha m} & 0 & 0 \\ 0 & -\frac{\eta\theta}{mq(\hat{x};\cdot)} & -\frac{\eta}{\lambda m q(\hat{x};\cdot)} & -\frac{\alpha\theta}{q(\hat{x};\cdot)} \\ -\frac{\tilde{x}}{q(\tilde{x};\cdot)} & \frac{1}{2\pi}+\lambda p_1 & p_1 & 0 \\ \frac{\tilde{x}}{q(\tilde{x};\cdot)} & \theta\lambda p_2 & \frac{1}{2\pi}+p_2 & -\frac{\hat{x}}{q(\hat{x};\cdot)} \end{bmatrix} \begin{bmatrix} d\tilde{x} \\ dN_d \\ dN_a \\ d\hat{x} \end{bmatrix}$$

$$= \begin{bmatrix} dl + \frac{\eta N_d}{\alpha^2 m} d\alpha + \frac{\eta N_d}{\alpha m^2} dm \\ dA - (\frac{\partial R}{\partial l}dl + \frac{\partial R}{\partial \alpha}d\alpha + \frac{\partial R}{\partial \theta}d\theta + \frac{\partial R}{\partial \lambda}d\lambda + \frac{\partial R}{\partial m}dm + \frac{\partial R}{\partial u}du + \frac{\partial R}{\partial y}dy) \\ -\int_0^{\tilde{x}} \frac{t}{q^2(t;\cdot)}(\frac{\partial q}{\partial l}dl + \frac{\partial q}{\partial \alpha}d\alpha + \frac{\partial q}{\partial \theta}d\theta + \frac{\partial q}{\partial \lambda}d\lambda + \frac{\partial q}{\partial m}dm + \frac{\partial q}{\partial u}du + \frac{\partial q}{\partial y}dy)dt \\ -\int_{\tilde{x}}^{\hat{x}} \frac{t}{q^2(t;\cdot)}(\frac{\partial q}{\partial l}dl + \frac{\partial q}{\partial \alpha}d\alpha + \frac{\partial q}{\partial \theta}d\theta + \frac{\partial q}{\partial \lambda}d\lambda + \frac{\partial q}{\partial m}dm + \frac{\partial q}{\partial u}du + \frac{\partial q}{\partial y}dy)dt \end{bmatrix} \tag{Appendix C.1}$$

For the presentation of the statics, it is useful to have a simplified expression for the determinate of the left hand side of equation (Appendix C.1). That determinate will be refered to as F where

$$F \equiv -\frac{1}{4\pi^2\alpha\lambda m^2 q^2(\hat{x};\cdot)q(\tilde{x};\cdot)}(4\pi^2\eta^2\hat{x}\tilde{x}$$
$$+\alpha^2\theta\lambda m^2 q(\tilde{x};\cdot)q(\hat{x};\cdot)(2\pi(p_2+\lambda p_1)+4\pi^2\lambda(1-\theta)p_1p_2+1)$$
$$+2\pi\eta\alpha m\hat{x}q(\tilde{x};\cdot)(2\pi\lambda p_1(1-\theta)+1)+2\pi\eta\alpha\theta\lambda m\tilde{x}q(\hat{x};\cdot)(2\pi(p_1+p_2)+1)) < 0$$

The simplified right hand side of equation (Appendix C.1) will be presented for each claim as well as the statics obtained.

Appendix C.3.1. Claim 3.1

Set all derivatives on the right hand side of equation (Appendix C.1) not related to l equal to 0. Use p_1 and p_2 as defined above and notice from

Appendix Appendix C.1 that

$$\frac{dq}{dl} = \begin{cases} 0 & \text{if } 0 < x < \tilde{x} \\ \dfrac{\lambda m}{\eta}\dfrac{dq}{dN_a} & \text{if } x > \tilde{x} \end{cases}$$

The right hand side of equation (Appendix C.1) is then

$$\begin{bmatrix} \frac{1}{\alpha(1-\theta)} \\ \frac{1}{q(\hat{x};\cdot)} \\ 0 \\ -\frac{\lambda m}{\eta}p_2 \end{bmatrix} dl$$

Solving gives the statics:

$$\frac{d\tilde{x}}{dl} = -\frac{1}{4\pi^2\alpha\lambda(1-\theta)mq(\hat{x};\cdot)^2F}(2\pi\eta\hat{x}+\alpha\lambda\theta mq(\hat{x};\cdot)(2\pi(p_2+\lambda p_1)+1)) > 0$$

$$\begin{aligned}\frac{dN_d}{dl} = -&\frac{1}{2\pi\eta\alpha\lambda(1-\theta)mq(\tilde{x};\cdot)q(\hat{x};\cdot)^2F}(2\pi\eta^2\hat{x}\tilde{x}+\eta\alpha\lambda\theta m\tilde{x}q(\hat{x};\cdot)(2\pi(p_1+p_2)+1) \\ &+ 2\pi\alpha\lambda mp_1q(\tilde{x};\cdot)(1-\theta)(\eta\hat{x}+\alpha\lambda\theta mp_2q(\hat{x};\cdot))) > 0\end{aligned}$$

$$\begin{aligned}\frac{dN_a}{dl} = &\frac{1}{2\pi\eta\alpha(1-\theta)mq(\tilde{x};\cdot)q(\hat{x};\cdot)^2F}(2\pi\eta^2\tilde{x}\hat{x} \\ &+ \alpha(1-\theta)mq(\tilde{x};\cdot)(\eta\hat{x}+\alpha\lambda\theta mp_2q(\hat{x};\cdot))(2\pi\lambda p_1+1) \\ &+ \eta\alpha\theta m\tilde{x}q(\hat{x};\cdot)(2\pi\lambda(p_1+p_2)+1)) < 0\end{aligned}$$

$$\frac{d\hat{x}}{dl} = \frac{1}{4\pi^2\alpha\lambda(1-\theta)mq(\tilde{x};\cdot)q(\hat{x};\cdot)F}(2\pi\eta\tilde{x}(\lambda-1)+\alpha\lambda(1-\theta)mq(\tilde{x};\cdot)(2\pi\lambda p_1+1)) < 0$$

$$\frac{dN}{dl} = \frac{1}{2\pi\eta\alpha\lambda(1-\theta)mq(\tilde{x};\cdot)q(\hat{x};\cdot)^2F}(2\pi\eta^2\tilde{x}\hat{x}(\lambda-1)$$
$$+\alpha\lambda(1-\theta)mq(\tilde{x};\cdot)(\eta\hat{x}+\alpha\lambda\theta mp_2q(\hat{x};\cdot))(2\pi p_1(\lambda-1)+1)$$
$$+2\pi\eta\alpha\lambda\theta m\tilde{x}q(\hat{x};\cdot)(p_1+p_2)(\lambda-1)) < 0$$

Appendix C.3.2. Claim 3.2

Set all derivatives on the right hand side of equation (Appendix C.1) not related to m equal to 0. Use p_1 and p_2 as defined above and notice from Appendix Appendix C.1 that

$$\frac{dq}{dm} = -\frac{N_a+\lambda N_d}{m}\frac{dq}{dN_a} \qquad \text{if } 0 < x < \tilde{x}$$
$$\frac{dq}{dm} = -\frac{N_a+\theta\lambda N_d}{m}\frac{dq}{dN_a} \qquad \text{if } \tilde{x} < x$$

So with a substitution from Appendix Appendix C.2, the right hand side of equation (Appendix C.1) can be expressed as

$$\begin{bmatrix} \frac{\eta N_d}{\alpha m^2} \\ -\frac{\eta}{q(\hat{x};\cdot)}(\frac{\theta N_d}{m^2}+\frac{N_a}{\lambda m^2}) \\ \frac{N_a+\lambda N_d}{m}p_1 \\ \frac{N_a+\theta\lambda N_d}{m}p_2 \end{bmatrix} dm$$

Solving gives the statics:

$$\frac{d\tilde{x}}{dm} = -\frac{\eta}{4\pi^2\alpha\lambda m^3q^2(\hat{x};\cdot)F}(2\pi\eta N_d\hat{x}+\alpha\theta\lambda mq(\hat{x};\cdot)(N_d-2\pi(p_1N_a-p_2N_d))) \gtrless 0$$

$$\frac{dN_d}{dm} = -\frac{1}{2\pi\alpha\lambda m^3 q(\tilde{x};\cdot)q^2(\hat{x};\cdot)F}(2\pi\eta^2 N_d\hat{x}\tilde{x} + 2\pi\eta\alpha\lambda(1-\theta)mN_d p_1\hat{x}q(\tilde{x};\cdot)$$
$$+ \eta\alpha\theta\lambda m N_d\tilde{x}q(\hat{x};\cdot)(2\pi(p_1+p_2)+1)$$
$$+ \alpha^2\lambda\theta m^2 p_1 q(\tilde{x};\cdot)q(\hat{x};\cdot)(N_a + \lambda N_d + 2\pi\lambda p_2 N_d(1-\theta))) > 0$$

$$\frac{dN_a}{dm} = -\frac{1}{2\pi\alpha\lambda m^3 q(\tilde{x};\cdot)q^2(\hat{x};\cdot)F}(2\pi\eta^2 N_a\hat{x}\tilde{x} + \eta\alpha\theta\lambda m\tilde{x}q(\hat{x};\cdot)(2\pi N_a(p_1+p_2) - N_d)$$
$$+ \alpha m q(\tilde{x};\cdot)(\eta\hat{x} + \alpha\theta\lambda m p_2 q(\hat{x};\cdot))(N_a + \theta\lambda N_d + 2\pi\lambda p_1 N_a(1-\theta))) \gtrless 0$$

$$\frac{d\hat{x}}{dm} = -\frac{\eta}{4\pi^2\alpha\lambda m^3 q(\tilde{x};\cdot)q(\hat{x};\cdot)F}(2\pi\eta\tilde{x}(2N_d + (N_a - N_d)(\theta\lambda + 2\pi(p_1+p_2)(\theta\lambda - 1)))$$
$$+ \alpha m q(\tilde{x};\cdot)(N_d + \theta\lambda N_a) + 2\pi\alpha\lambda m p_1 q(\tilde{x};\cdot)(N_d - \theta N_a + \theta\lambda(N_a - N_d))$$
$$+2\pi\alpha m p_2 q(\tilde{x};\cdot)(N_a - N_d)(\theta\lambda - 1) + 4\pi^2\alpha\lambda m p_1 p_2 q(\tilde{x};\cdot)(N_a - N_d)(1-\theta)(\theta\lambda - 1)) \gtrless 0$$

$$\frac{dN}{dm} = -\frac{1}{4\pi^2\alpha\lambda m^3 q(\tilde{x};\cdot)q^2(\hat{x};\cdot)F}(4\pi^2\eta^2 N_d\hat{x}\tilde{x} + 4\pi^2\eta\alpha\lambda m p_1 N_d\hat{x}q(\tilde{x};\cdot)(1-\theta)$$
$$+2\pi\eta\tilde{x}q(\hat{x};\cdot)(2\eta N_d + \eta\theta\lambda(N_a - N_d) + \alpha\theta\lambda m N_d + 2\pi(p_1+p_2)(\eta(N_a - N_d)(\theta\lambda - 1)$$
$$+\alpha\theta\lambda m N_d)) + \alpha m q(\tilde{x};\cdot)q(\hat{x};\cdot)(\eta N_d + \eta\theta\lambda N_a + 2\pi(\eta(N_a - N_d)(\lambda^2\theta p_1 + p_2(\theta\lambda - 1))$$
$$+\lambda\eta p_1(N_d - \theta N_a) + \alpha\theta\lambda m p_1(N_a + \lambda N_d)) + 4\pi^2\lambda p_1 p_2(1-\theta)(\eta(N_a - N_d)(\theta\lambda - 1)$$
$$+ \alpha\theta\lambda m N_d))) \gtrless 0$$

Appendix C.3.3. Claim 3.3

Set all derivatives on the right hand side of equation (Appendix C.1) not related to λ equal to 0. Use p_1 and p_2 as defined above and notice from

Appendix Appendix C.1 that

$$\frac{dq}{d\lambda} = -\frac{N_a}{\lambda}\frac{dq}{dN_a}$$

The right hand side of equation (Appendix C.1) is then

$$\begin{bmatrix} 0 \\ -\frac{\eta N_a}{\lambda^2 mq(\hat{x};\cdot)} \\ \frac{N_a}{\lambda}p_1 \\ \frac{N_a}{\lambda}p_2 \end{bmatrix} d\lambda$$

Solving gives the statics:

$$\begin{aligned}
\frac{d\tilde{x}}{d\lambda} &= \frac{\eta\theta N_a p_1}{2\pi\lambda mq(\hat{x};\cdot)F} < 0 \\
\frac{dN_d}{d\lambda} &= -\frac{\alpha\theta N_a p_1}{2\pi\lambda q(\hat{x};\cdot)F} > 0 \\
\frac{dN_a}{d\lambda} &= \frac{-N_a}{2\pi\alpha\lambda^2 m^2 q(\tilde{x};\cdot)q^2(\hat{x};\cdot)F}((\eta\hat{x} + \alpha\theta\lambda m p_2 q(\hat{x};\cdot))(2\pi\eta\tilde{x} \\
&\quad +\alpha mq(\tilde{x};\cdot)(2\pi\lambda p_1(1-\theta)+1)) + 2\pi\eta\alpha\theta\lambda m p_1\tilde{x}q(\hat{x};\cdot)) > 0 \\
\frac{d\hat{x}}{d\lambda} &= -\frac{\eta N_a}{4\pi^2\alpha\lambda^2 m^2 q(\tilde{x};\cdot)q(\hat{x};\cdot)F}(2\pi\eta\tilde{x} + \alpha mq(\tilde{x};\cdot)(2\pi\lambda p_1(1-\theta)+1)) > 0 \\
\frac{dN}{d\lambda} &= \frac{dN_d}{d\lambda} + \frac{dN_a}{d\lambda} > 0
\end{aligned}$$

Appendix C.3.4. Claim 3.4

Set all derivatives on the right hand side of equation (Appendix C.1) not related to θ equal to 0. Use p_1 and p_2 as defined above and notice from Appendix Appendix C.1 that

$$\frac{dq}{d\theta} > 0 \qquad \text{if } \tilde{x} < x$$

While it is possible to put $\frac{dq}{d\theta}$ in terms of variables already present, it is no more illuminating than defining $p_3 \equiv \int_{\tilde{x}}^{\hat{x}} \frac{t}{q^2(t;\cdot)} \frac{\partial q}{\partial \theta} dt > 0$. So with a substitution from Appendix Appendix C.2, the right hand side of equation (Appendix C.1) can be expressed as

$$\begin{bmatrix} 0 \\ \frac{\eta N_d + \alpha m}{m q(\hat{x};\cdot)} \\ 0 \\ -p_3 \end{bmatrix} d\bar{\theta}$$

Solving gives the statics:

$$\begin{aligned}
\frac{d\tilde{x}}{d\theta} &= \frac{\eta p_1}{\alpha m^2 q^2(\hat{x};\cdot)F}(\eta N_d \hat{x} + \alpha m(\hat{x} + \theta p_3 q(\hat{x};\cdot))) < 0 \\
\frac{dN_d}{d\theta} &= -\frac{p_1}{m q^2(\hat{x};\cdot)F}(\eta N_d \hat{x} + \alpha m(\hat{x} + \theta p_3 q(\hat{x};\cdot))) > 0 \\
\frac{dN_a}{d\theta} &= \frac{(\eta N_d \hat{x} + \alpha m(\hat{x} + \theta p_3 q(\hat{x};\cdot)))(2\pi\eta\tilde{x} + \alpha m(1 + 2\pi\lambda p_1) q(\tilde{x};\cdot))}{2\pi\alpha m^2 q(\tilde{x};\cdot) q^2(\hat{x};\cdot)F} < 0 \\
\frac{d\hat{x}}{d\theta} &= \frac{1}{4\pi^2 \alpha\lambda m^2 q(\hat{x};\cdot) q(\tilde{x};\cdot)}(\lambda(\alpha m + \eta N_d)(\alpha m(2\pi\lambda p_1 + 2\pi p_2(1 + 2\pi(1-\theta)\lambda p_1) + 1) q(\tilde{x};\cdot) \\
&\quad + 2\pi\eta\tilde{x}(2\pi(p_1 + p_2) + 1)) - 2\pi\eta p_3(\alpha(1 + 2\pi(1-\theta)\lambda p_1) m q(\tilde{x};\cdot) + 2\pi\eta\tilde{x})) \gtrless 0 \\
\frac{dN}{d\theta} &= \frac{(\alpha m q(\tilde{x};\cdot)(2\pi(\lambda - 1)p_1 + 1) + 2\pi\eta\tilde{x})(\alpha m(\theta p_3 q(\hat{x};\cdot) + \hat{x}) + \eta N_d \hat{x})}{2\pi\alpha m^2 q^2(\hat{x};\cdot) q(\tilde{x};\cdot)} > 0
\end{aligned}$$

Appendix C.3.5. Claim 3.5

Using the derivative related to A, the right hand side of equation (Appendix C.1) is then

$$\begin{bmatrix} 0 \\ dA \\ 0 \\ 0 \end{bmatrix}$$

Solving gives the statics:

$$
\begin{aligned}
\frac{d\tilde{x}}{dA} &= \frac{\eta p_1 \hat{x}}{\alpha m q(\hat{x};\cdot)F} < 0 \\
\frac{dN_d}{dA} &= -\frac{p_1 \hat{x}}{q(\hat{x};\cdot)F} > 0 \\
\frac{dN_a}{dA} &= \frac{\hat{x}}{2\pi\alpha m q(\tilde{x};\cdot)q(\hat{x};\cdot)F}(2\pi\eta\tilde{x} + \alpha m q(\tilde{x};\cdot)(2\pi\lambda p_1 + 1)) < 0 \\
\frac{d\hat{x}}{dA} &= \frac{1}{4\pi^2\alpha m q(\tilde{x};\cdot)F}(2\pi\eta\tilde{x}(2\pi(p_1+p_2)+1) \\
&\qquad + \alpha m q(\tilde{x};\cdot)(2\pi(p_2+\lambda p_1) + 4\pi^2\lambda(1-\theta)p_1 p_2 + 1)) < 0 \\
\frac{dN}{dA} &= \frac{dN_d}{dA} + \frac{dN_a}{dA} \\
&= \frac{\hat{x}}{2\pi\alpha m q(\hat{x};\cdot)q(\tilde{x};\cdot)F}(2\pi\eta\tilde{x} + \alpha m q(\tilde{x};\cdot)(2\pi p_1(\lambda-1)+1)) < 0
\end{aligned}
$$